Lecture 3. **Laplace integral transformation. Transfer functions; the formal properties of transfer function ACS**

To begin with, we introduce a complex variable $s = c + j\omega$, which represents a point in the complex plane.

Laplace transformation is performed as follows:

$$X(s) = \int_0^\infty e^{-st} x(t) dt \, .$$

Here *x(t)* is original of a signal, *X(s)* is an image of signal.

Direct Laplace transformation is denoted as $L\{x(t)\} = X(s)$, inverse Laplace transformation is denoted as $L^{-1}\{X(s)\} = x(t)$, where *L* and L^{-1} are direct Laplace and inverse Laplace operators correspondingly.

3.1 Main properties of Laplace transformation

1. *Linearity.* For any constants *α* and *β* Laplace transform of the sum of original signals is equal to the sum of original signals transforms:

$$L\{\alpha x_1(t) + \beta x_2(t)\} = \alpha L\{x_1(t)\} + \beta L\{x_2(t)\}.$$

2. *Differentiation property.* Differentiation of original signal *x(t)* at nonzero initial conditions is replaced by *s* in the power of differentiation order times signal transform *X(s):*

$$L\{\dot{x}(t)\} = sX(s) - x(0), \text{ where } X(s) = L\{x(t)\}, \; x(0) = \lim_{t \to +0} x(t);$$

$$L\{\ddot{x}(t)\} = s^2 X(s) - sx(0) - \dot{x}(0).$$

$$L(x^{(n)}(t)\} = s^n X(s) - [s^{n-1}x(0) + s^{n-2}\dot{x}(0) + s^{n-3}\ddot{x}(0) + \ldots + x^{(n-1)}(0)],$$

where $x^{(k)}(0) = \lim_{t \to +0} x^{(k)}(t)$, *k = 0,1,2, ..., n-1,*

In a more general form (at zero initial conditions):

$$x(0) = \dot{x}(0) = \ddot{x}(0) = \ldots = x^{(n-1)}(0) = 0.$$

Then we have $L\{x^{(n)}(t)\} = s^n X(s)$.

3. *Integration property.* Integration of original signal is replaced by s in the power of integration order divided by signal transform *X(s):*

$$L\{\int_0^t x(t) dt\} = \frac{1}{s}[X(s)].$$

4. *Shift theorem.* For any positive number τ transform of a signal with time shift $x(t-\tau)$ is equal to transform of the signal without time shift times *exp* in the power of $(-s\tau)$:

$$L\{x(t-\tau)\} = e^{-s\tau} L\{x(t)\} = e^{-s\tau} X(s).$$

5. *Convolution theorem (theorem of multiplying images).* Let original signals $x_1(t)$, $x_2(t)$ and their corresponding transforms $X_1(s)$, $X_2(s)$ be given. Then the product of the signal transforms is equal in time domain to the definite integral of the product of original signals where one of them is taken with positive time shift τ:

$$X_1(s)X_2(s) = \int_0^t x_1(\tau)x_2(t-\tau)d\tau = \int_0^t x_2(\tau)x_1(t-\tau)d\tau \,.$$

Right-hand side of this equation is called *convolution* of $x_1(t)$ and $x_2(t)$ and is denoted as $x_1(t) * x_2(t)$.

6. *Limit theorem.* Let $x(t)$ be an original signal, $X(s)$ is its transform. Then

$$x(0) = \lim_{s\to+\infty} sX(s)\,.$$

Even more, if the limit $x(\infty) = \lim_{t\to+\infty} x(t)$ exists, then $x(\infty) = \lim_{s\to 0} sX(s)$.

In future we will consider Laplace transformation at zero initial conditions, i.e.

$$X(0) = X'(0) = X''(0) = ... = X^{(n-1)}(0) = 0\,.$$

An example. Using [3, 5] our knowledge about Laplace operator let us rewrite education (2.1) using transforms:

$$a_0 s^n X_{out}(s) + a_1 s^{n-1} X_{out}(s) + ... + a_n X_{out}(s) =$$

$$= b_0 s^m X_{in}(s) + b_1 s^{m-1} X_{in}(s) + ... + b_m X_{in}(s) \quad . \qquad (1b)$$

Here we can see how exactly Laplace transformation helps us simplify rather complicated differential and integral equations.

3.2 Transfer functions. The formal properties of real system transfer function

The ratio of the disturbance operator to the proper operator is called *transfer function*.

Firstly, factor out $X_{out}(s)$ and $X_{in}(S)$ in (2.20) make after brackets. Then take the ratio:

$$\frac{X_{out}(s)}{X_{in}(s)} = \frac{b_0 s^m + b_1 s^{m-1} + ... + b_m}{a_0 s^n + a_1 s^{n-1} + ... + a_n} = W(s)\,. \qquad (2)$$

The ratio (2) in control theory is called *transfer function* in Laplace transformation form. In words, *transfer function* is the ratio of output variable transform to input variable transform at zero initial conditions.

What about nonzero initial conditions? Transfer function is the ratio of two polynomials $W(s)=\frac{Q_1(s)}{Q_2(s)}$. Denominator *$Q_2(s)$* is called *characteristic polynomial*. It characterizes properties of dynamic system (2.1). The equation followed

$$Q_2(s)=0 \qquad \text{or equally} \qquad a_0 s^n + a_1 s^{n-1} + \ldots + a_n = 0 \qquad (3)$$

is called *characteristic equation of the system.* Its proper numbers characterize *proper motion* of the system. But under nonzero initial conditions the denominator can not be set equal to zero, and proper numbers can not be found.

The numerator *$Q_1(s)$* of equation (2), when set equal to zero $Q_1(s)=0$, characterizes *forced motion* of the system. The roots of the education $Q_1(s)=0$ are usually called *noughts.* The roots of the equation (3) are usually called *poles.*

After all, we can rewrite output variable as:

$$W(s)=\frac{Q_1(s)}{Q_2(s)}=\frac{X_{out}(s)}{X_{in}(s)}\text{; hence, } X_{out}(s)=\frac{Q_1(s)}{Q_2(s)}X_{in}(s) \text{ or } X_{out}(s)=W(s)X_{in}(s). \qquad (4)$$

What will we have if in *transfer function W(s)* of a dynamic system we have the same multiplicands? For example:

$W(s)=\frac{(s+1)}{(s+1)}=1=const,$ that means that no dynamic is present. So you should be very careful with formula reduction.

Formal properties of a real system transfer function

As a rule, for a system transfer function in Laplace form the following conditions are held:

1) Coefficients transfer function *W(s)* fare real numbers.

2) Power of the numerator is less than of the denominator $m \le n$; mathematically, transfer function is a correct fraction. Physically, it means that the system is aftereffect, not the lead one.

3) The poles of the characteristic polynomial are located in the left half-plane. Physically, this fact means that free movement of the system asymptotically gets closer to zero as $t \to \infty$.

In modern control theory, systems under consideration are always real and can be described by differential equations. That is, signals and functions are continuous and differentiable.

3.3 Structuring dynamic system models

Complexity of a dynamic system is defined by the degree of the describing differential equation. Graphical representation of mathematical model of ACS as a joined links is considered as a structural scheme.

For simplification of analysis we decompose system into several parts; each part is then represented as a block diagram.

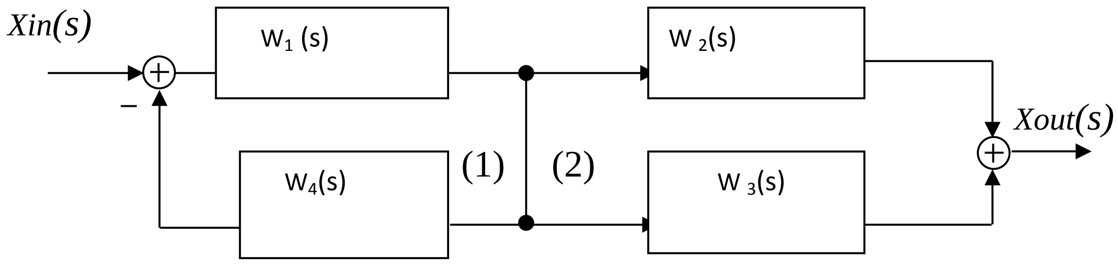


Fig. 2.2a. Fragment of a dynamic system structure

The system consists of structural elements that are represented as rectangles with definitions of input and output variables.

Here are some properties of structural elements:

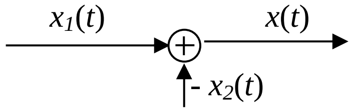
1) The elements supposed to have some unidirectional action. Hence all disturbances are applied to inputs.

2) Each element of a structural scheme is supposed to have only one input and only one output (keep in mind that, by definition, *W(s)* is a ratio of one signal to the other).

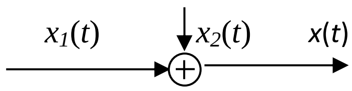
3) As a rule, negative feedback is used for real dynamic systems (positive feedback is used extremely rare).

Plus sign "+" defines junction where algebraic summing of signal appears.

Example 2.2. Consider the following examples:

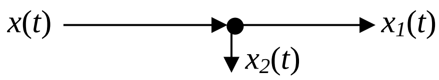


Here output signal is equal $x(t) = x_1(t) - x_2(t)$.



Here output signal is equal $x(t) = x_1(t) + x_2(t)$.

The following shows the symbol for branch point:



Note that the elements of *the structural scheme theory and the graph theory are identical.*

Thus, definition of structural schemes is represented by transfer functions and links equations.

Consider the fragment of the dynamic system shown in fig. 2.2a. The task is to make a mathematical formulation of the system and recognize its dynamic properties